Research of the Stress-Strain State of Conical Shell Under the Action of Local Load Based on the Non-Classical Theory

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ABSTRACT: The version of a refined theory of stress-strain computation of a conical shell under axially symmetric load is considered in the paper. The required displacements of the shell are represented in the form of polynomials in the normal coordinate, made for the middle surface two degrees higher with respect to the Kirchhoff – Love classical theory. Based on the equations of the three-dimensional theory of elasticity and the Lagrange variation principle, the equilibrium equations in displacements and the corresponding boundary conditions are obtained. The solution of laid down boundary value problem is carried out by the successive application of finite-difference and matrix sweep methods. Examples of calculating an isotropic conical shell rigidly restrained at two edges are given using a computer program. A significant contribution of additional transverse normal stresses in the boundary zone to the general stress state of the shell is shown. The influence of size of the local loading band on the stress state of the shell is studied. The results can be used to assess the strength and durability of structural elements in engineering facilities for various purposes.

KEYWORDS: conical shell, local load, improved theory, finite-difference method, matrix sweep method, stress-strain-state “boundary layer”, transverse normal stresses.

INTRODUCTION

Currently, designs like plates and shells are used extremely widely in key industries: in aircraft, rocket, instrument engineering, shipbuilding, etc. Engineering calculations of plates and shells rest on the results of the Kirchhoff-Love plate theory, which is based on the hypothesis of straight normals, which made it possible to represent a three-dimensional plate in two-dimensional form. However, when determining the stress-strain state (SSS) of shells, especially in the zones of joints (flanged, welded), local loading, alternating loadings, as well as structural elements made of heterogeneous materials, the classical theory does not provide reasonable compliance with a practice. To eliminate the shortcomings of the classical theory, it was necessary to take into account the transverse shear deformation and extension (compression).

One of the approaches to deriving a mathematically sound theory is the first-order shear deformation theory (FSDT) or the Timoshenko-Reissner theory [1, 2]. It says that the transverse straight lines normal to the mid-surface remain straight after deformation but will not be normal to the median surface. The FSDT theory requires the use of correction shear factors, influencing the accuracy of the calculation results. To address the shortcomings of the classical theory and the FSDT, Reddy, Touratier, Kant and Swaminathan used the Higher-Order Shear Deformation Plate Theory (HSDT) to study layered composite plates and shells. However, their assumptions do not satisfy the natural boundary conditions on their surfaces [3-6]. As a result, the error associated with the generalized Kirchhoff force disappeared in the shear theory, but such a shortcoming appeared as the impossibility of taking into account the self-balanced components of boundary forces.

In creating an approximate theory of shells without the Kirchhoff – Love hypotheses, the method of direct asymptotic integration of three-dimensional equations of the theory of elasticity is widely used. This method was developed by I.I. Vorovich, A.L. Goldenweiser and their staff [7-9]. In both series of works, the physically obvious property of the SSS of thin shells is clearly revealed, which is divided into internal and boundary properties. In the asymptotic method, this makes it possible to construct two iterative processes of integration of the differential equations based on the theory of elasticity for the case when the integration domain is sufficiently narrow. The first of these processes allows us to construct inner integrals with a given asymptotic accuracy. The second process determines rapidly the changing boundary integrals localized near the edges or other lines of
distortion of the total shell SSS and constituting the so-called “boundary layer”. Within the variational asymptotic method, using a specially derived approximating function with polynomials, V. Firsanov developed an improved theory for determining the stress-strain state in rectangular, round plates and cylindrical shells of constant and variable thickness near rigidly and elastically fixed boundaries [10-12]. This mathematical apparatus is used in paper to derive the main stress-strain state of a round plate with an asymmetric variable thickness [11]. The analysis showed that a slight change of plate thickness variability may result in significantly increased stiffness, strength and weight sophistication compared to a plate of constant thickness. It has been established that the additional stress-strain state near the fixed boundary significantly contributes to the overall stress state. Asymptotic methods are the main methods for transforming the equations of the elasticity theory in the works of L. Aghalovyan, A. Green, Yu. Dimitrienko and other scientists [13-18].

Another approach is proposed in the works of V.V. Vasiliev and S.A. Lurie [19, 20]. The desired displacements are approximated by polynomials in the coordinate normal to the middle plane of the shell and relate to each other the number of summands in these expansions in the tangential and transverse directions. The peculiarity of this approach is that the shell deformations are found using geometric relations, the tangential stresses are determined from the relations of Hooke’s law and transversal stresses are derived by direct integration of the equilibrium equations of the three-dimensional theory of elasticity. It has been established that significant additional local stresses appear already with an increase of approximating polynomials by one or two orders of magnitude with respect to the classical theory. Based on this approach, in papers [21–24] an improved theory of calculating the SSS for cylindrical and spherical shells is constructed, according to which additional local stresses turn out to be of the same order with the maximum stresses of the main SSS. Within the improved theory for a cylindrical shell, the paper describes a frequency equation that allows determining high tones of free vibrations not described by the classical theory [23].

This paper studies the stress-strain state of a conical shell under local axisymmetric loading within the framework of the approach presented in [19-24]. This paper is aimed at developing the boundary-value problem of mathematical model of a refined theory of conical shell and researching the contribution of additional boundary normal stresses to the general stress state. The basic equations of the improved shell theory are derived using the Lagrange variational principle and decomposition of the desired axis displacement along the normal to the mid-surface of the plate into polynomials of two degrees higher than in the Kirchhoff-Love plate theory. This technique allows us to consider not only thin shells, but also shells of medium thickness.

THE GOVERNING EQUATIONS

The conical shell related to the orthogonal system of curvilinear coordinates $x$, $\varphi$, $\xi$ (fig.1.) is considered. Here, $x$ is a coordinate on the cone generatrix, $\varphi$ is angle between certain axial plane and the reference plane, and the $\xi$ axis is directed along the outer normal to the middle surface.

![Figure 1. Conical shell.](image)
We suppose that the shell is under the effect of radial axisymmetric local load distributed over internal surface of the shell by law of:

\[
q_x = \begin{cases} 
0 & x_i^0 \leq x < x_i \\
q(x) & x_i \leq x \leq x_i^0 \\
0 & x_i^0 < x \leq x_i^0 
\end{cases}
\]  

(1)

where \( x_i^0, x_i \) are coordinates of the beginning and the end of cone along the generatrix.

The desired elastic displacements are represented as follows:

\[
u(x, \xi) = u_0(x) + u_1(x) \xi + u_2(x) \xi^2 + u_3(x) \frac{\xi^3}{3!};
\]

\[
w(x, \xi) = w_0(x) + w_1(x) \xi + w_2(x) \xi^2 + w_3(x) \frac{\xi^3}{2!}.
\]

(2)

Expansion (2) corresponds to a two-fold increase in the degree of polynomials approximating the desired displacements along the normal coordinate with respect to the classical theory.

Substituting expansions (2) into the geometric and physical equations of the theory of elasticity, after the transformations we obtain the system of equations:

\[
\sum_{n=0}^{2} \left( K_{00}^{n} + K_{01}^{n} \frac{d}{dx} + K_{11}^{n} \frac{d^2}{dx^2} \right) u_n + \sum_{n=0}^{2} \left( K_{00}^{n} + K_{01}^{n} \frac{d}{dx} + K_{11}^{n} \frac{d^2}{dx^2} \right) w_n = 0, \quad i = 1, 4.
\]

\[
\sum_{n=0}^{3} \left( K_{00}^{n} + K_{01}^{n} \frac{d}{dx} \right) u_n + \sum_{n=0}^{3} \left( K_{00}^{n} + K_{01}^{n} \frac{d}{dx} + K_{11}^{n} \frac{d^2}{dx^2} \right) w_n = K^{i} q_i, \quad i = 5, 7.
\]

(3)

The corresponding boundary conditions on the edges at \( x = \text{const} \) are represented as follows:

- for rigidly restrained edge \( u_n = w_n = 0, m = 0, 3, n = 0, 2 \).

(4)

- for free edge

\[
\sum_{n=0}^{2} G_{00}^{n} u_n + \sum_{n=0}^{2} G_{01}^{n} w_n = 0, \quad i = 1, 4,
\]

\[
\sum_{n=0}^{3} G_{00}^{n} u_n + \sum_{n=0}^{3} G_{01}^{n} \frac{dw_n}{dx} = 0, \quad i = 5, 7.
\]

(5)

- for hinge-supported edge

\[
\sum_{n=0}^{2} G_{00}^{n} + G_{01}^{n} \frac{d}{dx} u_n = 0, \quad i = 1, 4,
\]

\[
w_n = 0, \quad n = 0, 2.
\]

(6)

Here, \( K, G \) are variable coefficients depending on geometric parameters, elastic constants of the shell material and the \( x \) coordinate, and \( u_n, w_n \) are the expansion coefficients of the desired displacements in the expressions (2).

**THE SOLUTION OF THE FORMULATED BOUNDARY VALUE PROBLEM**

The system of ordinary equations (3) is solved by the finite-difference method. Derivatives of the first and the second orders are approximated by the central differences of the second order of accuracy, which allows us to obtain the following system:
\[
\sum_{n=0}^{2} \left( \frac{K_{11}^{(n)}}{s^2} - \frac{K_{11}^{(n)}}{2s} \right) u_{n}^{i-1} + \left( -\frac{2K_{11}^{(n)}}{s^2} + \frac{K_{01}^{(n)}}{2s} \right) u_{n}^{i} + \left( \frac{K_{11}^{(n)}}{s^2} + \frac{K_{11}^{(n)}}{2s} \right) u_{n}^{i+1} \right) \\
+ \sum_{n=0}^{2} \left( -\frac{K_{11}^{(n)}}{2s} w_{n}^{i-1} + \frac{K_{10}^{(n)}}{2s} w_{n}^{i} + \frac{K_{11}^{(n)}}{2s} w_{n}^{i+1} \right) = 0, \quad i = 1, 4,
\]

\[
\sum_{n=0}^{2} \left( -\frac{K_{11}^{(n)}}{2s} u_{n}^{i-1} + \frac{K_{01}^{(n)}}{2s} u_{n}^{i} + \frac{K_{11}^{(n)}}{2s} u_{n}^{i+1} \right) \\
+ \sum_{n=0}^{2} \left( \frac{K_{11}^{(n)}}{s^2} - \frac{K_{11}^{(n)}}{2s} \right) w_{n}^{i-1} + \left( -\frac{2K_{11}^{(n)}}{s^2} + \frac{K_{01}^{(n)}}{2s} \right) w_{n}^{i} + \left( \frac{K_{11}^{(n)}}{s^2} + \frac{K_{11}^{(n)}}{2s} \right) w_{n}^{i+1} \right) = K_{n}^{i} q_{n}, \quad i = \overline{5, 7}, \quad j = \overline{1, N-1},
\]
The system (8), taking into account boundary conditions (4-6), is solved by the matrix sweep method using the computer program. As a result, at the mesh points, shell movements are obtained, for approximations of which the splines are used. The shell deformations are determined from the geometric relations according to the theory of elasticity, the tangential stresses are found using the equation of the generalized Hooke's law. Transverse stresses are obtained by direct integration of the equilibrium equations of the three-dimensional theory of elasticity.

NUMERICAL ANALYSIS AND DISCUSSION

As an example of calculation, we consider the closed conical shell rigidly restrained on two edges , having following parameters: the angle of half of conicity , the beginning and the end along the shell generatrix, respectively , , the relative semi thickness , the Poisson ratio , the Young modulus . The shell is under effect of the following kinds of loads:

- a local load evenly distributed over a part of internal surface of shell

\[
q_e = \begin{cases} 
0 & \text{if } x_i^0 \leq x < x_1 \\
q_0 & \text{if } x_1 \leq x \leq x_2 \\
0 & \text{if } x_2 < x \leq x_2^0 
\end{cases}
\]  
(9)

- a local load linear distributed over a part of internal surface of shell
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\[
q_z = \begin{cases} 
0 & x_1 \leq x < x_1^0 \\
Q & x_1 \leq x \leq x_2 \\
0 & x_2 < x \leq x_2^0 
\end{cases}
\]

(10)

where \( x_1 = \frac{x_1^0 + x_1^0}{2} - \frac{\tau}{2} \), \( x_2 = \frac{x_2^0 + x_1^0}{2} + \frac{\tau}{2} \), \( \tau \) is a width of loading band.

In fig. 2 – 4 the results of calculation of SSS of the shell are shown. In fig. 2 and fig. 3 the results of calculation at \( \tau = 1/10x_1^0 \) for the case of load (9) are shown. Note that the «cl» abbreviation corresponds to the calculation data according to a classical theory.

![Figure 2](image1.png)

**Figure 2.** Change of the transverse stress along the shell length

![Figure 3](image2.png)

**Figure 3.** Change in deflections along the length of the shell

Analyzing the results, we can draw the following conclusions:

- there are rapidly damped additional stress states of the boundary-layer type near the rigidly fixed boundaries (Fig. 2);
- the maximum deflection along the generatrix of the shell is refined approximately at the level of 15% (Fig. 3).

Next, the effect of the size of the loading band on SSS of the shell is considered. We suppose that the shell is under effect of the load (10) at \( \tau = 1/5x_1^0 \) (fig. 4a), \( \tau = 2x_1^0 \) (fig. 4b), and \( \tau = 4x_1^0 \) (fig. 4c). An analysis of graphs in fig. 4 shows that:

- additional boundary-layer transverse stresses \( \sigma_{33} \) make up more than 40% of the maximum normal stresses \( \sigma_{11} \);
- in increasing a width of the loading band the normal stresses of the «boundary layer» are increased.
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**Figure 4.** Variation of stresses along the boundary $x = x^0$, thickness for various sizes of the loading band

**CONCLUSION**

Based on the results obtained, the following can be established:

- we have derived the boundary-value problem of the mathematical model of a refined theory of conical shell under local loading. The basic equations are derived using the Lagrange variational principle and the representation of two-order upward dislocation by polynomials in the normal coordinate with respect to the classical theory. The formulated boundary-value problem is solved by the consistent application of finite-difference and matrix sweep methods.

- near the areas of the stress state distortion, additional transverse stresses, which are neglected by the Kirchhoff-Love plate theory, have the same order as the maximum values of the main bending stress. This result allows us to reliably assess the strength and crack resistance of isotropic and composite structural elements in engineering facilities for various purposes.

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