Contact Mechanics for Soft Hemi elliptical Robotic Fingertip

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ABSTRACT: To increase the robustness and stability of grasps during manipulation, soft fingertips are preferred to use in the gripper of robotic and prosthetic hand. Establishing a relationship between imposed load and contact area is an important step to understand the governing mechanics behind it. A new analytical model for hemi-elliptical soft fingertip is established herein. This is performed by the use of nonlinear contact mechanics to have a relationship that relates the normal force with the radii of contact as a power-law equation. Experiments were conducted in order to verify this model. The soft fingertips used herein are made from three types of silicone rubber. The results indicated that the radii of contact ellipse are proportional to the imposed load raised to the power of γ that falls within (0 ≥ γ ≥ 1/3) and ellipticity parameter K. The magnitude of K is proportional to the ratio of curvature radii of hemi-elliptical shape raised to the power of α that falls within (1 ≥ α ≥ 0.636). The power indices α and γ are dependent on the material hardness as well as the geometry of fingertip. This new model incorporates Hamrock model for the linear elastic materials of elliptical contact with (α = 0.636) and Kao model for the non-linear elastic materials of circular contact with (K =1). Weighted residuals least-squares curve fitting was used for the purpose of fitting the experimental results. The experimental results supported the proposed theoretical prediction. In conclusion, the hemi-elliptical shape fingertips are better in contact properties as compared to the hemi-spherical shape fingertips.

KEYWORDS: Soft fingertip, Robotic hand, Non- linear elastic materials, Power-law contact theory, Ellipticity parameter.

INTRODUCTION

Soft fingertips in a multi-fingered hand are demonstrated higher interaction with the environment as compared with one that having hard fingertips [1,2]. The most significant features of the soft fingertip are the conformability of the object’s surface, the capability to dissipate repetitive strains and the capability of damping dynamic effects. Furthermore, the increase of contact area attributable to the deformation of tip causes the increase of the friction coefficient [3]. Realistic modeling of contact mechanics of soft fingertip plays a significant role in the stability of grasping along with the safe of handling object during manipulation. Good knowledge of contact characteristics reveals a proper estimation of grasping forces as well as the relationship that relates the normal force with the radius of contact. Considering these factors improve the robustness and the stability of grasps [4,5]. Different researchers tackled the contact characteristics soft fingertips via the use of various kinds of rubber materials and silicone [6,7]. Later, in a study demonstrated experimentally that the Hertzian theory of the linear elastic contacts cannot be used for the nonlinear elastic contacts, due to large deformations and nonlinearity of these materials [8]. Other researchers studied the human fingers and introduced the concept that the evolution of contact area related to the applied force [9]. Meanwhile, a group researchers developed an artificial finger based on experimental data of human finger [10]. N. Xydas and I. Kao expanded the Hertzian theory to include the nonlinear elastic deformation and verified their work experimentally utilizing artificial hemispherical soft fingertips (using silicone and rubber), their modeling showed that the contact radius is proportional to the normal force raised to the power of γ [11]. This theory is subsumed the linear elastic contact theory by Hertz. Kao and Yang followed the soft finger contacts theory, to derive equations that described the nonlinear stiffness behavior of a hemispherical soft finger [12]. For the anthropomorphic hemi-cylindrical soft fingers, derived a nonlinear contact-mechanics model of a hemi-cylindrical soft fingertip, and the results depicted that the half width of rectangular contact is related to the applied normal load as a power-law equation, within the power of γ that ranges 0 ≤ γ ≤ ½ [13]. A group researcher used an elastomeric material and derived the relation between the contact area (determined using ultrasonic signals) and the applied normal force of a hemispherical soft contact probe [14]. Elango and Marappan analyzed the effect of the deformation of soft fingertip during power grasping by the development of contact model [15].
The influence of soft thickness and material grades on the power law equation was investigated where a dimensionless parameter (Power Factor) was proposed, and it was found that the increase of soft thickness caused an increase in the value of exponent $\gamma$ [16]. For investigating the grasping of the very fragile objects, a group scientist proposed fingertip constructed from incompressible fluid covered by rubber [17]. Sun Hao examined the contact between ball and ball socket and found a critical dimensionless number ($f = 0.54$), which shifts the point contact to the line contact for the linear elastic materials [18]. In another study, the finite element analysis (FEA) to study the contact parameters of a hemispherical soft fingertip pressed against a non-flat surface and derived the power law model [19]. Literature review reveals that most of the researchers investigated the contact parameters of hemi-cylindrical and hemispherical soft fingertip. But, most of the practical application needs a better gripper model for the hemi-elliptical shape of fingertip. Hence, the main objective of the current work is to derive contact mechanics for the anthropomorphic hemi-elliptical soft fingertips, including nonlinear materials, through theoretical analysis and experimental validation using three types of silicone rubber with hemi-elliptical shape.

MODELING OF CONTACT MECHANICS

In this section, the linear elastic contact model is summarized; also, the derivation of the elliptical contact including nonlinear materials of the soft finger is introduced. Thus, more realistic of the soft fingertips will be represented.

Theory of Contact for A Linear Elastic Model

It is well-known that when two contacting surfaces are pressed against each other under load, they deformed and the mechanical contact is related to the applied load and the material’s hardness. The first analysis of mechanical contact was provided conducting experiments on glass, and the results revealed that the growth of the contact radius $a_i$ is related to the imposed loads $N$ as [20]:

$$a_i = CN^{1/3}$$

(1)

Where, $C$ is a proportional constant related to the geometry and material. For elliptical contacts, the contact area is described by an ellipse. Hamrock and Dowson extended the Hertzian theory to derive the main elliptical contact formulae to find the semi-major axis ($a$) and semi-minor axis ($b$) of the ellipse contact as [21]:

$$a = \lambda K^{1/3}C N^{1/3}$$
$$b = \lambda K^{-1/3}C N^{1/3}$$

(2)

Where, $\lambda$ is a constant depended on the hemi-elliptical shape, and $K$ is the ellipticity parameter (the ratio of the growth of semi-major axis to semi-minor axis of the ellipse contact (i.e $K = a/b$). The magnitude of $K$ is proportional to the ratio of the curvature radii of hemi-elliptical shape as [21]:

$$K = 1.0339 \left( \frac{R_i}{R_s} \right)^{0.636}$$

(3)

Modeling of Nonlinear Elastic Contact

The modeling of contact mechanics for nonlinearity elastic materials of hemispherical soft fingers was proposed [11]:

$$a_i = CN^{\gamma}$$

(4)

Where, $a_i$ is the circular contact radius, and $\gamma$ is the exponent of soft contacts within the range from (0) to (1/3) that depends on the fingertip material. The main objective in this section is to extend the modeling of the elliptical linear elastic contact in “Equation (2)” to include the nonlinear contact mechanics that represents more realistic soft fingertips. The 3-D constitutive relation of the incompressible nonlinear elastic materials is given as [22]:

$$\dot{\sigma}_{ij} = \left( \sigma / k_i \right)^n \partial \sigma_{ij} / \partial \sigma_{ij}$$

(5)
\[ \sigma_{ij} = \sigma_v \delta \left( \frac{\sigma_i}{k}, \delta_j \right) / \partial \sigma_{ij} \]  \quad (6)

The von Mises stress is:
\[ \sigma_v = \sqrt{3/2(S_{ij}S_{ij})} = \sqrt{3/2\left(\sigma_{ij} - \sigma_{kk} \delta_{ij} / 3\right)\left(\sigma_{ij} - \sigma_{kk} \delta_{ij} / 3\right)} \]  \quad (7)

The strain components are:
\[ \dot{\sigma}_j = \left( \dot{\sigma}_{ij} / \partial x_j + \dot{\sigma}_{ij} / \partial x_j \right) / 2 \]  \quad (8)

Where, \( k \) is the stress unit constant, \( n \) is the factor of strain-hardening (\( n \leq 1 \)), and \( u_i, u_j \) are the infinitesimal displacement in the \( i, j \) directions, respectively. The stress equilibrium necessitates that:
\[ \partial \sigma_{ij} / \partial x_j = 0 \]  \quad (9)

Consider the hemi-elliptical soft fingertip with radii \( R_x \) and \( R_y \) being pressed onto a flat rigid plane “Figure 1.a,” and taking the semi-axes \( a \) and \( b \) in the direction of the Cartesian coordinates \( x \) and \( y \) axes, respectively, the boundary conditions at the surface of the hemi-elliptical tip are:
\[ \sigma_x = 0 \quad \text{for} \quad x > a \]
\[ \sigma_y = 0 \quad \text{for} \quad y > b \quad \text{(no contact)} \]  \quad (10)

Where, \( a \) is the semi-major axis of elliptical contact, \( b \) is the semi-minor axis of elliptical contact, and \( \sigma_x, \sigma_y \) present the stresses in \( x \) and \( y \) direction, respectively. As shown in the “Figure 1.b,” the displacement of the hemi-elliptical surface \( u(\rho) \) in the contact zone due to the imposed load can be described as:
\[ u(\rho) = \delta - \left( R - \sqrt{R_x^2 - \rho^2} \right) \quad \text{for} \quad x < a, \quad y < b \quad \text{(in contact)} \]  \quad (11)

Where, \( \delta \) is the displacement in the contact zone at \( \rho = 0 \), and \( R \) is the reduced radius of curvature which can be described as [21]:
\[ 1/R = 1/R_x + 1/R_y \]  \quad (12)

Moreover, the force balance is required that:
\[ N = \int_A \sigma_{zz} \ dA \]  \quad (13)

Where, \( dA \) is the ellipse contact area, and “Equation (13)” can be written as:
\[ N = \int_0^{2\pi} \int_0^\rho \sigma_{zz} \rho \ d\rho \ d\theta \]  \quad (14)
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\[ N = \pi \int_{0}^{\frac{\pi}{2}} \sigma_{zz} \, d\left(\rho^2\right) \]  

(15)

Where, \( \sigma_{zz} \) is the stress component which is normal to the contact surface. The dimensionless variables are:

\[ \tilde{\rho} = \rho / \sqrt{ab} \quad , \quad \tilde{z} = z / \sqrt{ab} \quad , \quad \tilde{x} = x / \sqrt{ab} \quad , \quad \tilde{u} = u / R / ab \]  

(16)

Where, \( u \) is given by “Equation (11)”. Substituting \( \tilde{u} \) and \( \tilde{u} \) into “Equation (8)” yields:

\[ \dot{\sigma}_{zz} = \left(\sqrt{ab} / R\right) \dot{\sigma}_{zz} \]  

(17)

Where, \( \dot{\sigma}_{zz} = \left(\sqrt{ab} / R\right) \dot{\sigma}_{zz} \). From “Equation (5)”, after the substitution of \( \epsilon_{ij} \) in “Equation (17)” and \( \sigma_{zz} \) in “Equation (7)” one gets:

\[ \sigma_{ij} \sim \left(\sqrt{ab} / R\right)^{1/n} \sigma_{ij} \]  

(18)

Where “ \( \sim \) ” denotes that \( \sigma_{ij} \) is proportional \( \left(\sqrt{ab} / R\right)^{1/n} \sigma_{ij} \). Substituting “Equations (16) and (18)” into “Equation (15)” yields:

\[ N / \pi ab = \pi \int_{0}^{\frac{\pi}{2}} \sigma_{zz} d\left(\rho^2 / ab\right) = \left(\sqrt{ab} / R\right)^{1/n} \int_{0}^{\frac{\pi}{2}} \tilde{\sigma}_{zz} d\left(\tilde{\rho}^2\right) \]  

(19)

The integration of “Equation (19)” is dimensionless. By grouping all constant terms to obtain:

\[ N = C_1 a b \left(\sqrt{ab}\right)^{1/n} \]  

(20)

Substituting \( K = a / b \) gives:

\[ N = C_1 \left(a^2 / K\right) \left(a / K^{1/2}\right)^{1/n} \]  

(21)

Or,

\[ N = C_1 K b^{2} \left(K^{1/2} b\right)^{1/n} \]  

(22)

That leads to:

\[
\begin{align*}
    a &= C K^{1/2} N^{n(2n+1)} = C K^{1/2} N^{\gamma} \\
    b &= C K^{-1/2} N^{n(2n+1)} = C K^{-1/2} N^{\gamma}
\end{align*}
\]  

(23)

Where, \( K \) is the ellipticity parameter that depends on the ratio of curvature radii of hemi-elliptical shape, which can be obtained experimentally, and \( \gamma = n/2n+1 \) is the exponent of the applied load. Equation (23) is the new modeling of the nonlinear elastic contact that relates the normal force with the radii of ellipse contact as a power-law equation. For the linear elastic materials, the constant \( n \) is equal to 1. Equating “Equation (2)” with “Equation (23)” gives \( \lambda = K^{1/6} \). Thus, it’s the Hamrock contact model:

\[
\begin{align*}
    a &= C K^{1/2} N^{1/3} \\
    b &= C K^{-1/2} N^{1/3}
\end{align*}
\]  

(when \( n = 1 \))

Equation (24), which is in agreement with the Hamrock’s contact theory in “Equation (2)” is a special case of “Equation (23)”. Also, “Equation (23)” subsumes the Kao’s contact model for the hemispherical nonlinear elastic materials in “Equation (4)” with \( K = 1 \).
\[ a = b = C N^\gamma \]  

(25)

In general, \(0 \leq n \leq 1\); the range of the exponent in “Equation (23)” is \(0 \leq \gamma \leq 1/3\). If \(\gamma = 0\), this corresponds to the ideal case, where, the radii of contact are constant and independent of the applied load [22,23].

EXPERIMENTAL WORK

Silicone Rubber Selection

Three different silicone rubbers (G828, G-815, and G-801) were used as soft fingertips for the experimental verification. Table 1 shows some properties of these silicone rubbers. The soft fingertips, “As shown in Figure 2” are made as a hemi-elliptical shape with seven different curvature radius ratios \((R_x/R_y = 1, 1.25, 1.5, 1.75, 2, 2.25, \text{and } 2.5)\). Where, it is important to locate the ‘x’ and ‘y’ coordinates so that the following condition is fulfilled \(R_x/R_y \geq 1\). The values of \(R_x\) and \(R_y\) were calculated from “Equation (12)” with the reduced radius of curvature \(R = 6 \text{ mm}\), and the results of \(R_x\) and \(R_y\) are listed in “Table 2”.

Table 1. Properties of the used silicone rubbers

<table>
<thead>
<tr>
<th>Material Used</th>
<th>Hardness (shore A)</th>
<th>Mixing ratio</th>
<th>Tensile Strength (MPa)</th>
<th>Tear Strength (kN/m)</th>
<th>Elongation - break (%)</th>
<th>Curing Time (25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>G-828</td>
<td>28±1</td>
<td>100:3</td>
<td>4.2</td>
<td>26</td>
<td>420</td>
<td>2-3h</td>
</tr>
<tr>
<td>G-815</td>
<td>15±2</td>
<td>10:1</td>
<td>3.5</td>
<td>14</td>
<td>550</td>
<td>2-3 h</td>
</tr>
<tr>
<td>G-801</td>
<td>6±1</td>
<td>10:4</td>
<td>2.2</td>
<td>8</td>
<td>1000</td>
<td>2.5 h</td>
</tr>
</tbody>
</table>

Table 2. The values of hemi-elliptical radii

<table>
<thead>
<tr>
<th>(R_x/R_y)</th>
<th>1</th>
<th>1.2</th>
<th>1.5</th>
<th>1.75</th>
<th>2</th>
<th>2.25</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_x)</td>
<td>12</td>
<td>13.2</td>
<td>15</td>
<td>16.5</td>
<td>18</td>
<td>19.5</td>
<td>21</td>
</tr>
<tr>
<td>(R_y)</td>
<td>12</td>
<td>11</td>
<td>10</td>
<td>9.42</td>
<td>9</td>
<td>8.66</td>
<td>8.4</td>
</tr>
</tbody>
</table>

Figure 2. Three different silicone rubbers made as a hemi-elliptical shape with seven different curvature radius ratios.
Experimental Setup

The experimental setup, shown in “Figure 3.a”, was used for the investigation of the power-law in “Equation (23)” using different soft fingertip. Each fingertip was mounted on the vertical pole with a linear stage to press the soft fingertip on the solid flat surface, and the normal force generated was displayed on the readout panel. The fingertip's surface was coated via fine toner dust, to give a clear circumference of the finger imprint on the recording paper, and the contact area was measured directly from the finger imprint. The recording paper was placed on a plexiglas plate that used as a smooth surface for all experiments to avoid the distortion of fingertip imprints. All the fingertips used in the experiments have a circular contact area for the hemispherical fingertip and an ellipse contact area for the hemi-elliptical fingertip, as depicted in “Figure 3.b.”. The finger imprints were printed at a range of imposed load (0-100 N), and the radii of contact areas were measured with accuracy of ± 0.1 mm.

![Figure 3. (a) Experimental Setup, (b) Fingertips imprints](image)

Experimental Results

The hemispherical fingertip ($R_x/R_y = 1$) pressed on the flat solid surface with a range of normal force (0-100 N) and created a circular contact area. The radius of contact area was measured by using the imprints with a digital image processing technique and recorded for different normal forces. The mean values of the experimental results were estimated, as revealed in “Figure 4”, and the weighted residuals least-squares curve algorithm was used to fit the experimental data and evaluate the exponent $\gamma$ and constant $C$ for each type of silicone rubber. The results of this parametric are listed in “Table 3”.

![Figure 4. Experimental results for three types of silicone of a hemispherical fingertip ($R_x/R_y =1$) with the least-squares best curve fitting (LSBF)](image)

For the hemi-elliptical soft fingers with different curvature radius ratios of $R_x/R_y$, the experiments manifested that the shapes of the contact areas are an ellipse, and the growth of semi-major axis ($a$) and semi-minor axis ($b$)
of the contact ellipse was measured and recorded for each ratio of hemi-elliptical \((R_x/R_y)\), as elucidated in “Figures. 5, 6 and 7”. And, the least-square linear regression was applied to obtain the best fit line of elliptical parameter \((K=a/b)\) for each curvature radius ratio \((R_x/R_y)\). The results are listed in Table 3.

**Figure 5.** Experimental results of the growth of semi-major axis \(a\) and semi-minor axis \(b\) of hemi-elliptical silicone (G-828) with different ratios of radius \(R_x/R_y\) with the least-squares best line fitting (LSBF).

**Figure 6.** Experimental results of the growth of semi-major axis \(a\) and semi-minor axis \(b\) of hemi-elliptical silicone (G-815) with different ratios of radius \(R_x/R_y\) with the least-squares best line fitting (LSBF).
Figure 7. Experimental results of the growth of semi-major axis $a$ and semi-minor axis $b$ of hemi-elliptical silicone (G-801) with different ratios of radius $R_x/R_y$ with the least-squares best line fitting (LSBF).

Table 3. The values of elliptical parameter $K$ for each $(R_x/R_y)$

<table>
<thead>
<tr>
<th>$R_x/R_y$</th>
<th>G-828 $K=a/b$</th>
<th>G-815 $K=a/b$</th>
<th>G-801 $K=a/b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1.25</td>
<td>1.21</td>
<td>1.22</td>
<td>1.24</td>
</tr>
<tr>
<td>1.5</td>
<td>1.36</td>
<td>1.39</td>
<td>1.47</td>
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<td>1.75</td>
<td>1.52</td>
<td>1.55</td>
<td>1.7</td>
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<tr>
<td>2</td>
<td>1.65</td>
<td>1.73</td>
<td>1.85</td>
</tr>
<tr>
<td>2.25</td>
<td>1.79</td>
<td>1.89</td>
<td>2</td>
</tr>
<tr>
<td>2.5</td>
<td>1.9</td>
<td>2.05</td>
<td>2.19</td>
</tr>
</tbody>
</table>

The Elliptical Parameter $K$

The experimental results in Table 3 that produced the results illustrated in “Figure 8,” indicated that the elliptical parameter $K$ for each type of silicone is an exponential function of the curvature radius ratio for a hemi-elliptical fingertip and can be written as:

$$K = C_e \left( \frac{R_x}{R_y} \right)^{\alpha}$$

The constant parameters $C_e$ and $\alpha$ that depend on the geometry and material can be calculated by using the weighted least-squares algorithm, and the results are given in Table 4. It can be seen that the value of $\alpha$ for each type of silicone is within the range $1 \geq \alpha \geq 0.636$. Where, the Hamrock’s contact model for the linear elastic materials in “Equation (3)” is a spatial case from this formula with $\alpha = 0.636$. For the ideal case, the ratio of the growth radii of the contact ellipse ($K=a/b$) is independent of the material properties and it’s equal to the ratio of the hemi-elliptical geometry radius, thus $\alpha = 1$. 

![Graph showing elliptical parameter K vs. ratio of hemi-elliptical radii (Rx/Ry)]
Figure 8. The relationship between the ellipticity parameter $K$ and the ratio of hemi-elliptical radius for three types of hemi-elliptical silicone with the least-squares best curve fitting (LSBF)

Table 4. The experimental results of constant $C$, $C_e$ and the exponents $\gamma$, $\alpha$ for three types silicone.

<table>
<thead>
<tr>
<th>Type of silicone</th>
<th>$C$</th>
<th>$\gamma$</th>
<th>$C_e$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>G-828 (hard)</td>
<td>1.503</td>
<td>0.3026</td>
<td>1.02</td>
<td>0.6934</td>
</tr>
<tr>
<td>G-815 (middle)</td>
<td>2.2422</td>
<td>0.2694</td>
<td>1.016</td>
<td>0.7572</td>
</tr>
<tr>
<td>G-801 (soft)</td>
<td>3.1455</td>
<td>0.2253</td>
<td>1.0238</td>
<td>0.8474</td>
</tr>
</tbody>
</table>

DISCUSSIONS OF THE RESULTS AND CONTACT AREA

After conducting all the experiments of fingertips, the experimental results in “Fig. 4,” show that the radius of contact for a hemispherical fingertip $a_t$ is related to the applied load as a power-law equation, and the exponent $\gamma$ and constant $C$ for each type of silicone rubber are obtained in Table 4. It is evident that the exponent values of $\gamma$ for all three types of silicone fall within the adopted range in the theory $0 \leq \gamma \leq 1/3$. The results depicted that the softer materials resort to have smaller exponent values $\gamma$ with higher value of proportionality constant $C$. These results are perfectly in line with the models presented [11]. For hemi-elliptical soft fingertips with different curvature radius ratios of $R/R$, the contact area is an ellipse with semi-major axis $a$ and semi-minor axis $b$. The growth of semi-axis ($a$ and $b$) is a function of the normal load $N$ raised to power $\gamma$ (which presented in a hemispherical fingertip with the same of reduced radius curvature R) multiplied by a function of ellipticity parameter $K$ which is presented in “Equation (23)” Where, the elliptical parameter ($K=a/b$) for each curvature radii ratio ($R/R$) is obtained as revealed in “Figures 5, 6 and 7,” and the results listed in Table 3. Where, the softer material has a greater value of ellipticity parameter $K$.

The experimental results in Table 3 that produced the results illustrated in “Figure 8,” display that the ellipticity parameter $K$ is a function of ratio of a hemi-elliptical shape $R/R$, raised to power $\alpha$. The power law equation ($\alpha$) and the constant ($C_e$) for each type of silicone rubber were obtained by employing a weighted least-squares algorithm. The exponent values of $\alpha$ for all three types of silicone rubber fall within the adopted range in this theory $0.636 \leq \alpha \leq 1$. It is observed that the exponent value ($\alpha$) for harder materials tends to have smaller value. Also, it can be seen that the value of proportionality constant $C_e$ is very close form 1, that indicates the ellipticity parameter $K=a/b$ basically depends on $R/R$, and the effect of tip size is very small. In general, the experimental data of three types of silicone fallen within $0 \leq \gamma \leq 1/3$ and $1 \geq \alpha \geq 0.636$ is in agreement with the theoretical prediction of the upper bound $\gamma = 1/3$ and $\alpha = 0.636$ (which corresponds to the linear elastic materials) and $\gamma = 0$ and $\alpha = 1$ as a lower bound (for the ideal soft finger).

Normalization of the experimental curves can eliminate the constant $C$ from “Equation (23)”? and can be described as the following equation:

$$a/C = K^{1/2}N^{\gamma} \rightarrow a/C = \left(R_s/R_e\right)^{\alpha/2}N^{\gamma} \quad (27)$$

$$b/C = K^{-1/2}N^{\gamma} \rightarrow b/C = \left(R_s/R_e\right)^{-\alpha/2}N^{\gamma} \quad (28)$$

The results of the normalized curves for hemi-elliptical soft fingers with different curvature radius ratios of $R/R$, of three types of silicone are plotted in “Figure 9,” and “Figure 10,”.
Since the range of exponent $0 \leq \gamma \leq 1/3$, “Equations (27) and (28)” evinced that the growth rate of the ellipse radii of fingertip, which have larger exponent, is initially smaller than the softer fingertip with smaller $\gamma$ and then becomes larger. If two fingertips, having different exponents of $(\gamma_1, \alpha_1)$ and $(\gamma_2, \alpha_2)$, the change rate of the normalized elliptical contact radii related to the applied normal load can be derived as:

$$d \frac{a}{dN} = \left( \frac{R_x}{R_y} \right)^{\alpha_2} \frac{\gamma}{N^{-1}}$$  \hspace{1cm} (29)$$

$$d \frac{b}{dN} = \left( \frac{R_x}{R_y} \right)^{\alpha_2} \frac{\gamma}{N^{-1}}$$ \hspace{1cm} (30)$$

The intersection of the two fingertips rate leads to:

$$Nc(a) = \exp \left( \frac{\ln(\gamma_2 - \ln(\gamma_1)) + \left(\alpha_2 \ln\left(\frac{R_x}{R_y}\right) - \alpha_1 \ln\left(\frac{R_x}{R_y}\right)\right)}{\gamma_1 - \gamma_2} / 2 \right)$$ \hspace{1cm} (31)$$

$$Nc(b) = \exp \left( \frac{\ln(\gamma_2 - \ln(\gamma_1)) - \left(\alpha_2 \ln\left(\frac{R_x}{R_y}\right) - \alpha_1 \ln\left(\frac{R_x}{R_y}\right)\right)}{\gamma_1 - \gamma_2} / 2 \right)$$ \hspace{1cm} (32)$$

Where, $Nc(a)$ and $Nc(b)$ are the critical values of normal force, at which the change rate of the ellipse contact radii is swapped between the two soft fingertips 1 and 2. “Figure 11,” presents a comparison between two hemi-elliptical ($R_x/R_y = 2$) fingertip contacts, which have the values of exponents $\gamma_1 = 1/3$, $\alpha_1=0.636$ and $\gamma_2 = 0.2253$.  

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**Figure 9.** The normalized contact of semi-major axis $a$ related to the normal force.

**Figure 10.** The normalized contact of semi-minor axis $b$ related to the normal force.
\(\alpha_2=0.8474\). From “Equation (31) and (32),” it is found that \(N_c(a) = 0.0524\ N\) for semi-major axis \(a\) and \(N_c(b) = 0.01347\ N\) for semi-minor axis \(b\). Where, before this intersection \((N < N_c)\), the growth rate of the elliptical contact radii for the linear elastic fingertip \((\gamma_1 = 1/3)\) is smaller than the softer fingertip \((\gamma_2 = 0.2253)\) and after this intersection \((N > N_c)\), the growth rate of the linear elastic fingertip is greater than the softer fingertip. The initial rate of growth of the ideally soft fingertip is infinite and then immediately reduces to zero. As a result, the critical normal force \(N_c\) is very small for typical fingers and it occurs very quickly under applying the normal force.

The preceding analysis and results can be employed in design that involves elliptical contact as a hemi-elliptical soft fingertip under large deformation where the power-law equations, which are presented in “(23),” can be used in place of the Hamrock’s contact equation. In these cases, if the linear elastic materials are used in design, the exponent should be taken as \(\gamma = 1/3\) and \(a = 0.636\). In general, once the values of the exponents \((\gamma\) and \(\alpha)\) are determined for the material experimentally, it can be employed for the artificial finger design using the preceding analysis presented in this paper.

![Figure 11. Comparison between two hemi-elliptical contacts with different values of exponents, for hard \((\gamma_1 = 1/3, \alpha_1 = 0.636)\) and soft \((\gamma_2 = 0.2253, \alpha_2 = 0.8474)\).](image)

The comparison between the hemispherical and hemi-elliptical fingertips indicates that the hemi-elliptical fingertips are more realistic for a human finger in terms of shape and ellipse imprint. Furthermore, since the width of finger design can be limited, the hemi-elliptical soft tip can be controlled easily by altering the ratio of hemi-elliptical radius to obtain larger contact area, whereas the hemispherical tip is limited due to the symmetry of axis. Finally, it was found that the hemi-elliptical tip size is smaller than hemispherical tip under the same contact properties.

**CONCLUSION**

A new modeling of contact mechanics of hemi-elliptical soft fingertip is proposed as a power-law equation that related to an ellipse contact area. The radii of ellipse contact are pro-rated to the applied normal load raised to the exponent \(\gamma\) within the range \(0 \leq \gamma \leq 1/3\) and ellipticity parameter \(K\). The results clearly manifested that the ellipticity parameter \(K\) is dependent on the material as well as curvature ratio, where \(K\) is a function of the ratio of hemi-elliptical shape \(R/R_e\) raised to power \(\alpha\) within the range \(0.636 \leq \alpha \leq 1\). The general results subsumed the Hamrock and Dowson model for the linear elastic contact mechanics. The experimental data were fitted by using the weighted least-squares algorithm. It is clear that the experimental results are in agreement with the proposed theoretical prediction. Finally, it can be seen that the hemi-elliptical structures are preferable over hemispherical ones for the robotic hand application, where it likes a human finger and it can be controlled easily by altering the ratio of semielliptical radius to obtain larger contact area.

**REFERENCES**


Nomenclature

The terms used in this work are:

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Definition</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>Semi-major axis of the ellipse contact area.</td>
<td>mm</td>
</tr>
<tr>
<td>a_s</td>
<td>Radius of the circular contact area.</td>
<td>mm</td>
</tr>
<tr>
<td>b</td>
<td>Semi-minor axis of the ellipse contact area.</td>
<td>mm</td>
</tr>
<tr>
<td>K</td>
<td>Ratio of the growth of semi-major axis to semi-minor axis of the contact ellipse.</td>
<td>_</td>
</tr>
<tr>
<td>k_s</td>
<td>Constant with stress units.</td>
<td>_</td>
</tr>
<tr>
<td>Ν</td>
<td>Applied normal load.</td>
<td>N</td>
</tr>
<tr>
<td>n</td>
<td>The exponent of stress that depends on the material.</td>
<td>_</td>
</tr>
<tr>
<td>R</td>
<td>The reduced radius of curvature.</td>
<td>mm</td>
</tr>
<tr>
<td>R_x</td>
<td>The radius of curvature in the ‘x’ direction.</td>
<td>mm</td>
</tr>
<tr>
<td>R_y</td>
<td>The radius of curvature in the ‘y’ direction.</td>
<td>mm</td>
</tr>
<tr>
<td>u(ρ)</td>
<td>Displacement in the contact due to applied load.</td>
<td>mm</td>
</tr>
<tr>
<td>γ</td>
<td>Exponent of the soft contact, as a power-law equation.</td>
<td>_</td>
</tr>
<tr>
<td>a</td>
<td>Exponent of the elliptical parameter k, as a power-law equation.</td>
<td>_</td>
</tr>
<tr>
<td>λ</td>
<td>Proportional constant Hamrock’s of equation; λ =[(2/π)(1.003+(0.5968R_x/R_y))]^{1/3}.</td>
<td>_</td>
</tr>
<tr>
<td>ρ</td>
<td>The equivalent radius of the contact area ρ = \sqrt{ab} .</td>
<td>mm</td>
</tr>
<tr>
<td>δ</td>
<td>The displacement in the contact zone at ρ = 0.</td>
<td>mm</td>
</tr>
<tr>
<td>σ_e</td>
<td>Von-Mises stress.</td>
<td>MPa</td>
</tr>
<tr>
<td>σ_p, ε_ij</td>
<td>Stress and strain components in the i, j directions.</td>
<td>MPa</td>
</tr>
<tr>
<td>σ_x, σ_y</td>
<td>Stresses in the x, y axes.</td>
<td>_</td>
</tr>
<tr>
<td>σ_{ij}, ε_{ij}</td>
<td>Nondimensionalized stress and strain components.</td>
<td>_</td>
</tr>
<tr>
<td>ρ, z, x</td>
<td>Nondimensionalized coordinates of ρ, z, x.</td>
<td>_</td>
</tr>
<tr>
<td>u_i</td>
<td>Nondimensionalized displacement in ith direction.</td>
<td>_</td>
</tr>
</tbody>
</table>