Stability Analysis and Assessment of Double Inverted Pendulum with LQR and LQG Controllers

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ABSTRACT

This paper presents the stability analysis and assessment of a double inverted pendulum on a cart with LQR and LQG control algorithms. First, the mathematical model of the system is presented. Then the LQR controller is designed to stabilize the inverted pendulum system without noise. The simulation results performed in the MATLAB environment show that the system has a good response. However, after adding noise, the system is no longer stable and oscillates around the equilibrium position. Therefore, the Kalman filter is applied in the LQG controller to filter out the noise. Simulations are performed and show the effectiveness of the Kalman filter in noise reduction. Finally, the LQG is applied to control the position of the cart.

KEYWORDS

Double pendulum system, LQR controller, LQG controller, Kalman filter.

INTRODUCTION

The double inverted pendulum is a nonlinear, multi-variable, and unstable system. Thus, in recent years, the inverted pendulum has become a typical model for verifying control theories. Setting up an inverted pendulum model is quite simple and low cost. The inverted pendulum system is a nonlinear system and has been considered by many studies where most of the system is linearized. The inverted pendulum system is stable when the two pendulums are in a vertical position and there are no oscillations. Stabilizing the inverted pendulum system with classical controllers is a difficult task because it is a nonlinear system with three degrees of freedom but only one control input signal. The balance of the pendulum is controlled by the reciprocating motion (left or right direction) of the mobile robot. To solve such problems with a non-linear time-varying system, there are alternatives such as real-time computer simulations of these equations or linearization. However, it also has its own shortcomings due to its principle, the system is loop unstable and highly non-linear; cause the pendulum to fall rapidly whenever the system is simulated due to the failure of standard linear techniques to model the non-linear dynamics of the system.

Many methods have been proposed to control the inverted pendulum, such as traditional PID control [1], fuzzy control [2], genetic algorithm optimizing control [3], and linear quadratic regulator (LQR) control [4]. In [5], a Linear Quadratic-Regulator (LQR) and a robust control technique for controlling the linearized system of an inverted pendulum model are presented and compared. In [6], a simple multi-PD controller designed on the theory of pole placement and its performance is compared with the Linear Quadratic Regulator controller using MATLAB and Simulink. Two control methods are proposed in [7], an innovative double PID control method and a modern LQR (linear quadratic regulator) control method. Dynamic performance and steady-state performance are investigated and compared of the two controllers. In [8], Linear Quadratic Regulator (LQR) and Pole Placement control strategy are used for solving tracking square wave and stabilization of pendulum around the upright position. The simulation shows that the performance of LQR is better than the pole
placement control. In [9], Linear Quadratic Gaussian (LQG) control strategy is used for solving the servo problem and stabilization of the angle of the pendulum rod. B. Agarwal and M. Bhandari [10] develop and implement Composite Nonlinear Feedback (CNF) control law for an inverted pendulum subject to actuator saturation. In [11], the comparative performance analysis of LQR and the state feedback pole-placement controller for the swing up and stabilization of the inverted pendulum system is presented. Shalaby et al. [12] propose an optimal time-varying linear quadratic Gaussian controller (TV-LQG) for the stabilization of an inverted pendulum. The TV-LQG utilizes a sigma-point Kalman filter (SPKF) and a linear quadratic regulator with a prescribed degree of stability.

In this paper, the control algorithms LQR and LQG are used for stabilizing the double inverted pendulum. In the LQR algorithm, the nonlinear system is linearized around the equilibrium point. LQR is a full state feedback controller that requires knowing all the variables. This is difficult to achieve in practice. The actual pendulum is also affected by noise (systemic noise and measurement noise). In the presence of noise, the LQR controller is unstable, so a Kalman filter is applied and an LQG controller is designed to reduce noise. LQG controller shows good performance even in the presence of noise. LQG algorithm is applied to control the position of the vehicle. The state of the system quickly stabilizes when the position of the vehicle is changed.

MATHEMATICAL MODEL OF THE INVERTED PENDULUM SYSTEM

The inverted pendulum model includes the main components such as the car running on the slider, the lower pendulum bar attached to the car by a rotational joint, and is connected to the upper pendulum bar by a rotational joint (see Figure 1). Encoders are mounted at the rotational joints to determine the value of rotational angles.

The parameters used in the mathematical model of the double inverted pendulum include:

- $m_0$: mass of the vehicle
- $m_1$: mass of the lower pendulum bar
- $m_2$: mass of the upper pendulum bar
- $u$: the input control force acting on the vehicle
- $L_1$: the length of the lower pendulum bar
- $L_2$: the length of the upper pendulum bar
- $r$: horizontal displacement of the vehicle
- $\theta_1$: angle between the lower pendulum and the vertical
- $\theta_2$: angle between the upper pendulum and the vertical
- $J_1$: inertia moment of the lower pendulum
- $J_2$: inertia moment of the upper pendulum

Newton’s second law was used to construct the mathematical model of the double inverted pendulum. First, all the forces acting on the car and the pendulum bars are analyzed. Then establish the equations of motion in the X and Y directions as well as the moment equilibrium equations. Finally, we can obtain the result through approximation and linearization around the equilibrium point as follows [13]:

![Figure 1. The principal structure of double inverted pendulum](image-url)
\[
\begin{bmatrix}
\ddot{\theta}_1 \\
\ddot{\theta}_2
\end{bmatrix} = A^{-1}b
\]

in which:
\[
A = \begin{bmatrix}
\frac{m_0 + m_1 + m_2}{2} & \frac{m_1}{2} + m_2 & \frac{m_2 L_2}{2} \\
\frac{m_1}{2} + m_2 & \frac{m_1}{4} + m_2 & \frac{m_2 L_2}{4} \\
\frac{m_2 L_2}{2} & \frac{m_2 L_2}{4} & \frac{m_2^2}{4} + f_1
\end{bmatrix}
\]
\[
b = \begin{bmatrix}
\frac{u}{m_2 L_2 g \sin \theta_1} \\
\frac{m_2 L_2 g \sin \theta_2}{2}
\end{bmatrix}
\]

Substitute \(m_0 = 2.5 \text{ kg}, m_1 = 0.22 \text{ kg}, m_2 = 0.22 \text{ kg}, L_1 = 0.5 \text{ m}, L_2 = 0.5 \text{ m}\) into (1), we obtain:
\[
\ddot{\theta}_1 = 0.00061 \ddot{\theta}_1 + 0.1803 \dot{\theta}_2 + 0.3902 u \\
\ddot{\theta}_2 = -76.9906 \theta_1 + 67.3545 \theta_2 + 0.3344 u
\]

Denote \(x_1 = \dot{r}, x_2 = \dot{\theta}_1, x_3 = \dot{x}_1, x_4 = \dot{\theta}_2, x_5 = \dot{x}_2, x_6 = \dot{\theta}_2 = \dot{x}_5\), the state equation of the inverted pendulum is:
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.6224 & 0 & 0.1803 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 54.5719 & 0 & -25.6635 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix} +
\begin{bmatrix}
0 \\
0.3902 \\
0 \\
-1.0033 \\
0 \\
0.3344
\end{bmatrix} u
\]

The feedback control law that minimizes the value of the cost is \(u = -Kx\), where \(K\) is given by:
\[
K = R^{-1}B^T P
\]

with \(P\) is found by solving the continuous-time Riccati differential equation:
\[
PA + A^T P + Q - BPR^{-1}B^T P = 0
\]
The control signal makes the state variables zero and \( J \) reach the minimum value. Hence \( u(t) \) is called the optimal control signal.

\[ u(t) \]

**Figure 2.** Diagram of the LQR controller

DESIGN OF THE LQG CONTROLLER

The state equation of the system in the presence of is as follows:

\[
\begin{align*}
\dot{x} &= Ax + Bu + w(t) \\
y &= Cx + Du + v(t)
\end{align*}
\]

(8)

where \( w(t) \) is the system noise and \( v(t) \) is the measurement noise. Assume that noise is white Gaussian noise of zero mean and variance:

\[
E[ww^T] = Q_N, \quad E[vv^T] = R_N
\]

We use the Kalman filter to reduce the noise and combine it with the LQR controller:

\[ \text{LQG} = \text{LQR} + \text{Kalman Filter} \]

The Kalman filter:

\[
\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)) \\
\dot{\hat{y}}(t) &= C\hat{x}(t)
\end{align*}
\]

(9)

in which:

\[ L = \Pi C^T R_N^{-1} \]

(10)

with \( \Pi \) is found by solving the equation:

\[
\Pi A + \Pi A^T + \Pi C^T R_N^{-1} C \Pi + Q_N = 0
\]

(11)

Combining the LQR with the Kalman filter, we obtain an LQG controller with the state equation as in (12). The block diagram of the LQG controller is shown in Figure 8.

\[
\begin{align*}
u(t) &= -K\hat{x}(t) \\
\hat{y}(t) &= C\hat{x}(t) \\
\hat{x}(t) &= (A - LC - BK)\hat{x}(t) + Ly(t) \\
\dot{x} &= Ax + Bu + w(t) \\
y &= Cx(t) + v(t)
\end{align*}
\]

(12)

**Figure 8.** Block diagram of LQG controller
SIMULATION RESULTS AND DISCUSSION

The system under consideration and the LQR controller are modeled and simulated in MATLAB. The step response performance of the controller is shown from Figure 3 to Figure 5 and the time response specifications are given in Table 1, 2 and 3. The results show that the controller has been successfully designed and gives satisfactory response and performance (peak amplitudes of 0.017 and 0.018 rad with settling time of 10.54s and 10.62 for $\theta_1$ and $\theta_2$ respectively).

The simulation is conducted with the values of $Q_N$ and $R_N$ as follows:

$$Q_N = 10^{-12} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_N = 10^{-6} \begin{bmatrix} 2500 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The simulation results are shown in Figure 6 and Figure 7. It can be seen that the LQR controller is very sensitive to noise.

![Figure 3. Step response of LQR control for cart position](image)

![Figure 4. Step response of LQR control for pendulum angles](image)
Table 1. Summary of the performance characteristics for cart position

<table>
<thead>
<tr>
<th>Time response specifications</th>
<th>LQR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Settling Time $T_s$</td>
<td>10.1 s</td>
</tr>
<tr>
<td>Rise Time $T_r$</td>
<td>3.66 s</td>
</tr>
<tr>
<td>Peak amplitude</td>
<td>1.032</td>
</tr>
<tr>
<td>Steady-state error $ess$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2. Summary of the performance characteristics for pendulum angle $\theta_1$

<table>
<thead>
<tr>
<th>Time response specifications</th>
<th>LQR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Settling Time $T_s$</td>
<td>10.54 s</td>
</tr>
<tr>
<td>Peak amplitude</td>
<td>0.01732</td>
</tr>
<tr>
<td>Steady-state error $ess$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3. Summary of the performance characteristics for pendulum angle $\theta_2$

<table>
<thead>
<tr>
<th>Time response specifications</th>
<th>LQR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Settling Time $T_s$</td>
<td>10.62 s</td>
</tr>
<tr>
<td>Peak amplitude</td>
<td>-0.0182</td>
</tr>
<tr>
<td>Steady-state error $ess$</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 5. Step response with LQR control for control signal

Figure 6. Step response of LQR control with noise for cart position
Figure 7. Step response of LQR control with noise for pendulum angles

Figure 9. Step response of LQG control for cart position

Figure 10. Step response of LQR control for pendulum angle $\theta_1$
The simulation results of LQG are shown from Figure 9 to Figure 12. The results show that the noise is almost completely suppressed. Although the "cart position" noise is 50 times larger than the pendulum angles noise, the filtering results for the cart's position are extremely good. We increase the noise of the rotation angles by 10 times. The results in Figure 13 show that the filter is still working very well. However, the pendulums oscillate slightly around the equilibrium position.
Figure 14 shows the results when applying the LQG algorithm to control the position of the vehicle. The position of the vehicle is changed to position 0.5 m at t = 10s, then to position 0.3m at t = 15s and back to 0 at t = 20s. The results show that the position of the car is very well controlled. The state of the system quickly stabilizes when the position of the vehicle is changed.

CONCLUSION

In this paper, LQR and LQG controllers have been used to control the double inverted pendulum. The LQR has a good performance when there is no noise in the system and measurement process. In the presence of noise, the LQR controller is unstable, so a Kalman filter is applied to the LQG controller to reduce noise. LQG controller shows good performance even in the presence of noise. The state of the system quickly stabilizes when the position of the vehicle is changed.

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REFERENCES


