Vibrations Analysis of Functionally Graded Beams at Different Boundary Conditions

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ABSTRACT

Functionally graded materials (FGM) are very important and modern applications in engineering due to the unique properties of these materials. In this work, the vibrations of FGM beams are studied at eight different boundary conditions such as F=F, SS=SS, C=C, C=SS, C=F, F=SS, C=SL, and F–SL. The exact equations of natural frequencies of FGM beams are derived for that different boundary conditions. A good agreement between present results and the results of previous literature. At all the different boundary conditions, the same behavior observed for natural frequencies at all modes when the physical properties of the beam are changing such as the thickness, the length, the density of material or ceramic, the elasticity of material or ceramic, and the ratio of volume fraction. When the ratio of volume fraction increases, the natural frequencies decrease. As the dimensionless ratio (L/h) increases, the natural frequencies will be decreased. When the dimensionless ratio (Em/Ec) increases, the natural frequencies increase too, but that frequencies decrease when the dimensionless ratio (ρm/ρc) increases.

KEYWORDS

Vibrations of beam, FGM material, Natural frequencies.

INTRODUCTION

Functionally graded materials (FGMs) are considered as asymmetrical microscopic compounds and are made of mixed materials of metals and ceramics. It is widely used in machinery, microelectronics, industrial applications, nuclear engineering, and spacecraft [1]. There are many applications of FGM beam structures in engineering. The vibration characteristics for this kind of beams is important to study the behavior of more complex structures that are exposed to similar conditions, the researchers studied the vibration of FGM beams using different procedures and theories. Avcara M. and Hazim H. [2] investigated the free vibration of the composed beam, (FGMs). According to the Rayleigh beam theory, the equation motion of the functionally graded beam was derived. A comparison was made with the results available for the homogeneous beam in order to validate the results. Zakaria I. et al [3] investigated the forced and free vibration of FGM beam, the refined method was used to develop the transverse shear strain and stretching across the FGM beam theory. Yang L. et al. [4] developed a mathematical model to compute the behavior of natural frequencies of FGM beams.


The effect of coating thickness, beam slenderness, and material parameters was considered. Irwan K. et al. [13] Investigated a two-node FGM beam using Timoshenko theory and finally, Xian-Fang Li [14] used the higher-
order shear deformation theory to analyze the vibration of FGM beams. A single fourth-order partial differential equation with constant coefficients was driven. In this work, the exact equations for calculating the natural frequencies of most conventional boundary conditions for FGM beams will be found. Also, the effects of some geometrical and material properties of FGM beam such as the thickness, the length, volume fraction ratio, the density, and the elasticity modulus of metal and ceramics on the behavior of the natural frequencies of the FGM beams will be investigated.

METHODOLOGY

In this study, consider a ceramic-metal as a functionally graded beam (FGM). Figure (1) shows the geometry of the FGM beam. The material properties across the beam thickness are varying based on volume fraction ratio.

![Figure 1. Geometry of FGM beam.](image)

The material properties can be obtained, according to the rule of the mixture as follow:

\[ P = P_m V_m + P_c V_c \]  \hspace{1cm} (1)

And,

\[ V_m + V_c = 1 \]  \hspace{1cm} (2)

The volume fraction of ceramic associating to law power is:

\[ V_c = \left( \frac{z}{h} + \frac{1}{2} \right)^r, \quad 0 \leq r \leq \infty \]  \hspace{1cm} (3)

From the above equation, the material properties of the FGM beam across the thickness can be written as follow:

\[ E(z) = (E_c - E_m) \left( \frac{z}{h} + \frac{1}{2} \right)^r + E_m \]  \hspace{1cm} (4)

\[ \rho(z) = (\rho_c - \rho_m) \left( \frac{z}{h} + \frac{1}{2} \right)^r + \rho_m \]  \hspace{1cm} (5)

The displacements of any arbitrary point of the FGM beam can be expressed by the Kirchoff-Love hypothesis as follow [2,7,15]:

\[ u(x, z, t) = u_o(x, t) + z\theta \]  \hspace{1cm} (6)

\[ w(x, z, t) = w_o(x, t) \]  \hspace{1cm} (7)

The normal and shear strains are obtained as follows:

\[ \varepsilon(x, z, t) = \frac{\partial u_o}{\partial x} + z \frac{\partial \theta}{\partial x} \]  \hspace{1cm} (8)

\[ \gamma_{xz} = \theta + \frac{\partial w_o}{\partial x} \]  \hspace{1cm} (9)

By the Rayleigh beam theory, the shear strain, \( \gamma_{xz} \) will be neglected. Thus, the normal strain is obtained as follow:
\[ \varepsilon(x,z,t) = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} \]  
\[ (10) \]

The normal stress is:
\[ \sigma(x,z,t) = E(z) \cdot \varepsilon = E(z) \left( \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} \right) \]  
\[ (11) \]

The axial force is:
\[ N_x = \int_A \sigma_x dA = A_{11} \frac{\partial u_0}{\partial x} - B_{11} \frac{\partial^3 w_0}{\partial x^2} \]  
\[ (12) \]

The bending moment is:
\[ M_x = \int_A \sigma_x z dA = B_{11} \frac{\partial^2 u_0}{\partial x} - D_{11} \frac{\partial^3 w_0}{\partial x^2} \]  
\[ (13) \]

The transverse shear force is:
\[ Q_x = \frac{\partial M_x}{\partial x} = B_{11} \frac{\partial^2 u_0}{\partial x} - D_{11} \frac{\partial^3 w_0}{\partial x^3} \]  
\[ (14) \]

Where [7]:
\[ A_{11} = \int_{-h/2}^{h/2} E(z) \cdot dz = \frac{h}{1-v^2} \left[ \frac{(E_c-E_m)}{(n+1)} + E_m \right] \]  
\[ (15) \]
\[ B_{11} = \int_{-h/2}^{h/2} E(z) \cdot z \cdot dz = \frac{h^2}{1-v^2} \left[ \frac{(E_c-E_m)h^2}{2(n+1)(n+2)} \right] \]  
\[ (16) \]
\[ D_{11} = \int_{-h/2}^{h/2} E(z) \cdot z^2 \cdot dz = \frac{h^3}{1-v^2} \left[ \frac{(E_c-E_m)(n^2+n+2)}{(n+1)(n+2)(n+3)} + E_m \right] \]  
\[ (17) \]

The FGM beams equations are found by neglecting the axial inertia term, and by using Hamilton’s principle, as follows [2,16]:
\[ \delta u_0; \frac{\partial N_x}{\partial x} = 0 \]  
\[ (18) \]
\[ A_{11} \frac{\partial^2 u_0}{\partial x^2} - B_{11} \frac{\partial^3 w_0}{\partial x^2} = 0 \]  
\[ (19) \]
\[ \delta w; \frac{-\partial^2 M_x}{\partial x} + I_o \frac{\partial^2 w}{\partial t^2} = 0 \]  
\[ (20) \]
\[ \left( D_{11} - \frac{B_{11}^2}{A_{11}} \right) \frac{\partial^4 w}{\partial x^4} + I_o \frac{\partial^2 w}{\partial t^2} = 0 \]  
\[ (21) \]

Where:
\[ I_o = \int_{-h/2}^{h/2} \rho(z) \cdot dz = h \left[ \frac{(E_c-E_m)}{(n+1)} + \rho_m \right] \]  
\[ (22) \]

Assuming the general function form at arbitrary boundary conditions for FGM beam as follow:
\[ W(x, t) = X(x) \cdot T(t) \]  
\[ (23) \]

Where:
\[ X(x) = (C_1 \sin \beta_n x + C_2 \cos \beta_n x + C_3 \sinh \beta_n x + C_3 \cosh \beta_n x) \]  
\[ (24) \]
\[ T(t) = (A_{1n} \sin \omega_n t + A_{2n} \cos \omega_n t) \]  
\[ (25) \]

Substituting equation (23) in equation (21) get:
\[ \left( D_{11} - \frac{B_{11}^2}{A_{11}} \right) \beta_n^4 - I_o \omega_n^2 = 0 \]  
\[ (26) \]

Let:
\[ \Pi_{11} = D_{11} - \frac{B_{11}^2}{A_{11}} \]  

(27)

From equation (26), the general form of natural frequency equation of FGM beams at arbitrary boundary conditions is:

\[ \omega_n = \sqrt{\frac{\Pi_{11} B_n^2}{I_o}} \]  

(28)

Applying the boundary conditions of simply supported FGM beam at both ends in equation (24), the natural frequency equation will be as follow:

\[ \omega_n = \sqrt{\frac{n^4 \Pi_{11}}{16 L^4 I_o}} \]  

(29)

The boundary conditions applied in equation (24) for FGM beams clamped in both ends firstly and secondly free in both ends, the same natural frequency equation is got for these beams as follow:

\[ \omega_n = \sqrt{\frac{(2n+1)^4 \Pi_{11}}{16 L^4 I_o}} \]  

(30)

Also, the boundary conditions for the cantilever FGM beam are applied, the natural frequency of this beam is:

\[ \omega_n = \sqrt{\frac{(2n-1)^4 \Pi_{11}}{16 L^4 I_o}} \]  

(31)

The same natural frequency equations are getting when the boundary conditions are applied for FGM beam simply-support in one end and clamped at another end, and FGM beam simply-support in one end and free in another end as follow:

\[ \omega_n = \sqrt{\frac{(4n+1)^4 \Pi_{11}}{256 L^4 I_o}} \]  

(32)

Also, when applied the boundary conditions for FGM beam free in one end and sliding in other end and FGM beam clamped in one end and sliding in another end, the same natural frequency equations are getting as follow:

\[ \omega_n = \sqrt{\frac{(4n-1)^4 \Pi_{11}}{256 L^4 I_o}} \]  

(33)

RESULTS AND DISCUSSION

In this work, verification tests are done to validate from equations that are used. the FGM beam is chosen in this study made from Alumina and Aluminum.

The data of FGM beam are: \( E_c = 380 GPa \), \( \rho_c = 3960 kg/m^3 \), \( v_c = 0.3 \), \( E_m = 70 GPa \), \( \rho = 2702 kg/m^3 \), \( v_m = 0.3 \).

The dimensionless frequency is defined as follow:

\[ \Omega = \frac{\omega_n h}{L^2 \sqrt{\frac{\rho_c}{E_c} \frac{\rho_m}{E_m}}} \]  

(34)

Table (1) shows a comparison between the results from the present work and the result of ref. [7] for FGM beam at both ends are simply-supported. In that table, a constant difference (4.68%) is noted between the present results and the results of reference [7]. That difference comes from neglecting the Poisson's ratio effect in reference [7]. If ignoring Poisson's ratio effect in the present results, the difference will be 0%.

**Table 1.** Verification test for FGM beam simply-support at both ends.

| n  | First Dimensionless Natural Frequency | Second Dimensionless Natural Frequency | Third Dimensionless Natural Frequency |
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Table (2) shows a comparison between the results from the present work and the result of ref. [7] for FGM beam at both ends are free. In that table, a constant difference (3.96%) in the first mode, (4.70%) in the second mode, and (4.68%) in the third mode are noted between the present results and the results of reference [7]. That difference also comes from neglecting the effect of the Poisson ratio in reference [7]. If the effect of the Poisson ratio is ignoring in the present results, the difference will be 0%. The difference is acceptable between the present results and the results of reference [7].

Table 2. Verification test for FGM beam free at both ends.

<table>
<thead>
<tr>
<th>r</th>
<th>First Natural Frequency</th>
<th>Dimensionless</th>
<th>Second Natural Frequency</th>
<th>Dimensionless</th>
<th>Third Natural Frequency</th>
<th>Dimensionless</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>5.753</td>
<td>5.483</td>
<td>4.69%</td>
<td>23.011</td>
<td>21.933</td>
<td>4.68%</td>
</tr>
<tr>
<td>0.2</td>
<td>5.353</td>
<td>5.102</td>
<td>4.68%</td>
<td>21.410</td>
<td>20.408</td>
<td>4.68%</td>
</tr>
<tr>
<td>0.5</td>
<td>4.898</td>
<td>4.669</td>
<td>4.68%</td>
<td>19.593</td>
<td>18.676</td>
<td>4.68%</td>
</tr>
<tr>
<td>1</td>
<td>4.428</td>
<td>4.221</td>
<td>4.68%</td>
<td>17.713</td>
<td>16.884</td>
<td>4.68%</td>
</tr>
<tr>
<td>2</td>
<td>4.041</td>
<td>3.852</td>
<td>4.68%</td>
<td>16.164</td>
<td>15.407</td>
<td>4.68%</td>
</tr>
<tr>
<td>5</td>
<td>3.848</td>
<td>3.668</td>
<td>4.67%</td>
<td>15.391</td>
<td>14.670</td>
<td>4.68%</td>
</tr>
<tr>
<td>6</td>
<td>2.989</td>
<td>2.849</td>
<td>4.69%</td>
<td>11.956</td>
<td>11.396</td>
<td>4.69%</td>
</tr>
</tbody>
</table>

The variation of dimensionless natural frequencies with the variation of volume fraction ratio of the FGM beam is shown in Figure (2) at different boundary conditions. For all boundary conditions, the same behavior for natural frequencies is noticed. The natural frequencies decrease as the volume fraction ratio increase. The largest effect when the ratio between (0 to 1), dimensionless natural frequency decrease about 30% in that range but when ratio increase from (1 to 6), the frequencies decrease less than 10% in all boundary conditions.
Figure 2. The variation of natural frequencies with volume fraction ratio.

Figure 3 shows the variation of natural frequencies with the dimensionless ratio (L/h) at different boundary conditions. In all BCs, the frequencies decrease when the dimensionless ratio length to thickness increases. This relation is so clearly observed in all modes.

Figure 3. The variation of natural frequencies with dimensionless ratio (L/h).

The variation of natural frequencies with the variation in the dimensionless ratio (Em/Ec) at different BCs is shown in figure (4). The natural frequencies increase as the ratio increase, that behavior the same in all boundary conditions and all modes. The values of natural frequencies increase about 50% when the ratio increase from 0.1 to 1, but the frequency increase less than 20% when the ratio increase from 1 to 2 and from 2 to 3, etc.
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Figure 4. The variation of natural frequencies with dimensionless ratio ($E_m/E_c$).

Figure 5 shows the variation of natural frequencies with the variation in the dimensionless ratio ($\rho_m/\rho_c$) at different boundary conditions. When the ratio increases, the natural frequencies decrease at all boundary conditions and modes. The values of natural frequencies decrease about 30% when the ratio increase from 0.1 to 1, but the frequency decrease less than 15% when the ratio increase from 1 to 2 and from 2 to 3, etc.

Figure 5. The variation of natural frequencies with dimensionless ratio ($\rho_m/\rho_c$).

CONCLUSIONS

In this work, the exact equations of natural frequencies of FGM beams are derived for eight different boundary conditions as F-F, SS-SS, C-C, C-SL, C-F, F-SS, C-SS, and F-SL. A good agreement between present results and the results of previous literature. At all the different boundary conditions, the same behavior observed for natural frequencies at all modes when the physical properties of the system are changing such as the length of the beam, the thickness of the beam, the density of material or ceramic, the elasticity of material or ceramic, and the ratio of volume fraction. The natural frequencies decrease as the volume fraction ratio increase and decrease when the dimensionless ratio ($L/h$) increase. When the dimensionless ratio ($Em/Ec$) increases, the natural frequencies increase too, but that frequencies decrease when the dimensionless ratio ($pm/pc$) increases.

REFERENCES


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